Test 2 Computational Methods of Science, November 2022

Duration: 2 hours.
In front of the questions, one finds the points. The sum of the points plus 1 gives the end mark for this test. Criteria used for the grading are: demonstration of understanding, logical reasoning, correct use of terminology, correctness of results.

1. Consider on the interval [0,1] the differential equation

$$\frac{du}{dx} - \frac{d}{dx}\left((1+x^2)\frac{du}{dx}\right) = \exp(x^2) \tag{1}$$

with boundary conditions $u(0) = 1$ and $\frac{du}{dx}(1) = 2$.

   (a) (1.2 points) Make a finite volume discretization of this equation on an equidistant grid with mesh size h, where the boundary condition at $x = 0$ is applied at a grid point and $x = 1$ in the middle between two grid points.

   (b) (1.0 points) Replace the right-hand side function in (1) by zero. Show that for sufficiently small h, the associated discretization in part a will lead to a monotonous solution. If you where not able to find that discretization use $(u_{j+1} - u_{j-1})/4 - (1 + ((j + \frac{1}{2})h)^3)(u_{j+1} - u_j)/h + (1 + ((j - \frac{1}{2})h)^3)(u_j - u_{j-1})/h = 0$.

   (c) (1.0 points) Show that the discretization derived in part a, divided by h, is a second-order accurate approximation of the given differential equation away from the boundaries. If you were not able to derive a discretization there, you may also use the one in the previous part, but then, you also have to determine *to which* differential equation it converges (since the discretization given in part b is not a discretization of differential equation (1)).

Exam text continues at other side

2. Consider the following partial differential equation on $[0, 1]$:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - d^2 u$$

for $t > 0$, where c and d are constants. The boundary conditions are $u(0, t) = g(t)$, $u_x(1, t) = 0$. Furthermore, we have the initial conditions $u(x, 0) = f(x)$ and $u_t(x, 0) = 0$.

(a) (1.2 points) Give the difference equations and their initial conditions that result when applying a second-order accurate finite-difference discretization in space on an equidistant grid with grid size Δx to the given initial boundary value problem. Apply the boundary conditions at the grid points.

(b) (0.8 points)] Show how the difference equations of part a are written in vector form

$$\frac{d^2}{dt^2}\mathbf{u} = A\mathbf{u} + \mathbf{b}(t) \tag{2}$$

and define $\mathbf{u}$, A and $\mathbf{b}$. Also give the vector form of the initial conditions.

(c) (1.0 points) Use the difference/Fourier method to get an estimate of the eigenvalues of A. These eigenvalues will be real. You may use that $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$.

(d) (0.4 points) Transform the system of second-order differential equations in (2) to a system of first-order differential equations and write it in the form

$$\frac{d}{dt}\mathbf{w} = B\mathbf{w} + \mathbf{h}(t). \tag{3}$$

Define $\mathbf{w}$, B and $\mathbf{h}$. Don't forget the initial conditions!

(e) (1.0 points) Assume λ is an eigenvalue of the matrix A. Show that $\mu = \pm\sqrt{\lambda}$ is an eigenvalue of the matrix B in the previous part. Assume $-\alpha \leq \lambda \leq 0$ for some $\alpha > 0$. Make a sketch of the position of the eigenvalues μ in the complex plane. What is a good aproximation of α based on the Fourier eigenvalues found in part c. If you were not able to find an estimate of λ in part c, you may use as estimate $-c^2(\sin\theta/\Delta x)^2 - d^2$ where Δx is the mesh width in x-direction.

(f) Consider for the problem $y' = f(t, y)$ the time-integration method $w_{n+1} = w_n + \Delta t f(t_{n+1}, w_n + \Delta t f(t, w_n))$.

  i. (0.3 points) Show that the amplification factor is $\rho(z) = 1 + z + z^2$.

  ii. (0.2 points) Show that for z on the imaginary axis, say $z = iy$ with y real, $\rho(z) \leq 1$ for $y \in [-1, 1]$.

  iii. (0.3 points) Express the maximum allowed time step Δt in terms of Δx when we apply this time integration method to the first-order system in part d, where we set $c = 1$ and $d = 4$.

(g) i. (0.2 points) Consider the eigenfrequency mode $\mathbf{u}(t) = \mathbf{v}\exp(i2\pi f t)$ where $\mathbf{v}$ is an eigenvector of A and λ the associated eigenvalue. Substitute it in (2) with $\mathbf{b}(t) \equiv 0$ and determine how the frequency f is related to λ.

  ii. (0.2 points) Let $\mathbf{b}(t) = \mathbf{v}\exp(i\omega t)$, where $\mathbf{v}$ is an eigenvector of A. Show that there is a solution of the form $\alpha\mathbf{v}\exp(i\omega t)$ to (2) and determine α.

  iii. (0.2 points) For the solution in the last part, what happens to α if ω gets close to $\pm\sqrt{\lambda}$, where λ is the eigenvalue of A associated to $\mathbf{v}$.